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Properties of the Strict Chebyshev Solutions of the Linear Matrix Equation $AX+YB=C$

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In this paper, the properties of the strict Chebyshev solutions of the linear matrix equation $AX+YB=C$ are investigated, where A, B, and C are given matrices of dimensions $m \times r$, $s \times n$, and $m \times n$, respectively, $m > r$, $n > s$. © 1988 Academic Press, Inc.

1. Introduction

Let $\mathcal{M}_{mn}$ be the space of real $(m \times n)$-matrices. We consider the linear matrix equation

$$AX+YB=C, \tag{1}$$

where $A=(a_{ik}) \in \mathcal{M}_{mr}$, $B=(b_{lj}) \in \mathcal{M}_{sn}$, and $C=(c_{ij}) \in \mathcal{M}_{mn}$ are given and $X=(x_{kj}) \in \mathcal{M}_{rn}$, $Y=(y_{il}) \in \mathcal{M}_{ms}$. We can write (1) in the form $Mx=c$ with $M=(I_n \otimes A,\ B^T \otimes I_m)$ and appropriate definitions of the vectors x and c, where $\otimes$ denotes the Kronecker product. The equation (1) has a solution X and Y if and only if (see [3])

$$(I-AA^-)\,C(I-B^-B)=0, \tag{2}$$

where A^- is any g-inverse of A such that $AA^-A=A$. In this paper we assume that Condition (2) is not satisfied and we find a Chebyshev solution of (1). Matrices X_∞ and Y_∞ are a Chebyshev solution of (1) if

$$\|AX_\infty + Y_\infty B - C\|_\infty = \delta_\infty \equiv \min_{X,Y} \|AX+YB-C\|_\infty, \tag{3}$$

where

$$\|A\|_\infty = \max_{i,j} |a_{ij}|.$$

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The matrix $Z_\infty = AX_\infty + Y_\infty B$ which satisfies (3) is a Chebyshev approximation of C by matrices of the form $AX + YB$. In the general case, for arbitrary A and B, the Chebyshev approximation of C is not unique. The main purpose of this paper is to investigate the properties of the strict Chebyshev approximation Z_∞ of C, which is determined uniquely. However, the strict Chebyshev solution X_∞ and Y_∞ of Eq. (1) is not unique. The concept of strict Chebyshev solution of an overdetermined linear system, due to Rice, has been formulated in a constructive manner by him [8, 9] and by Descloux [5].

We assume in what follows that Condition (2) is not satisfied and that $m > r$, $n > s$.

2. Main Result

We now consider the linear matrix equation (1). We introduce some auxiliary notation and definitions.

a_i, b_i, c_i, d_i—the ith column of A^T, B, C, and C^T, respectively;
e_j—the jth column of the identity matrix of order n;

$$r_{ij}(X; Y) = \sum_{k=1}^{r} a_{ik} x_{kj} + \sum_{l=1}^{s} y_{il} b_{lj} - c_{ij};$$

$$\phi_j(Z) = \|Ze_j - c_j\|_\infty - \min_{h \in R^r} \|Ah + Ze_j - c_j\|_\infty,$$

where $Z \in \mathcal{M}_{mn}$.

Let the matrices X_p and Y_p be the l_p-solution of (1)

$$\|AX_p + Y_p B - C\|_p = \min_{X, Y} \|AX + YB - C\|_p, \qquad 1 < p < \infty,$$

where

$$\|A\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{r} |a_{ij}|^p \right)^{1/p}$$

We denote $Z_p = AX_p + Y_p B$ and $r_{ij}^{(p)} = r_{ij}(X_p; Y_p)$. The matrix Z_p is determined uniquely. We know that the matrices X_p and Y_p are the l_p-solution of (1) if and only if they satisfy Condition P (see [10]), which means that for $j = 1, ..., n$, the jth column of X_p is the l_p-solution of the linear system

$$Ax = c_j - Y_p b_j \tag{4}$$

and for $i=1, \ldots, m$, the ith column of Y_p^T is the l_p-solution of

$$B^T y = d_i - X_p^T a_i. \tag{5}$$

We introduce the following auxiliary definition.

DEFINITION. We say that X and Y, $X \in \mathcal{M}_{rn}$ and $Y \in \mathcal{M}_{ms}$, satisfy *Condition* TC if for $j=1, \ldots, n$, the jth column x_j of X is a Chebyshev solution of the linear system

$$Ax = c_j - Yb_j \tag{6}$$

and for $i=1, \ldots, m$, the ith column y_i of Y^T is a Chebyshev solution of the linear system

$$B^T y = d_i - X^T a_i. \tag{7}$$

If the vectors x_j and y_i ($j=1, \ldots, n$; $i=1, \ldots, m$) are strict Chebyshev solutions of the systems (6) and (7), respectively, then X and Y satisfy *Condition* TS.

Remark. If the systems (6) or (7) are consistent then their Chebyshev solutions are equal to their solutions.

Let

$$Z_\infty = AX_\infty + Y_\infty B = \lim_{p \to \infty} Z_p. \tag{8}$$

This limit exists and the matrices X_∞ and Y_∞ are a strict Chebyshev solution of (1) (see [5]). We denote

$$r_{ij}^{(\infty)} = r_{ij}(X_\infty; Y_\infty), \qquad X_\infty = (x_{kj}^{(\infty)}), \qquad Y_\infty = (y_{il}^{(\infty)}).$$

We have the following lemma (compare Theorem 3 in [11]).

LEMMA 1. *Let the matrices* X_∞ *and* Y_∞ *satisfy* (8). *Then they satisfy Condition* TC.

Proof. We consider the linear systems

$$Ax = c_j - Y_\infty b_j \qquad (j=1, \ldots, n), \tag{9}$$

$$B^T y = d_i - X_\infty^T a_i \qquad (i=1, \ldots, m). \tag{10}$$

Suppose that there exists j_1 $(1 \leqslant j_1 \leqslant n)$ such that the j_1th column of X_∞ is not a Chebyshev solution of (9) for $j = j_1$. Therefore there exists $\varepsilon > 0$ such that

$$\phi_{j_1}(Z_\infty) > \varepsilon.$$

Thus, by the lower semi-continuity of the function ϕ_j, there exists $\delta > 0$ such that

$$\phi_{j_1}(Z) > \varepsilon, \qquad \text{for} \quad Z \in \mathcal{M}_\delta,$$

where

$$\mathcal{M}_\delta = \{Z: Z \in \mathcal{M}_{mn}, \|Z - Z_\infty\|_\infty \leqslant \delta\}. \tag{11}$$

From (8) there exists p_0 such that $Z_p \in \mathcal{M}_\delta$ for $p > p_0$. Hence we obtain

$$\phi_{j_1}(Z_p) > \varepsilon, \qquad \text{for} \quad p > p_0. \tag{12}$$

By the properties of the l_p-norm we have

$$\|x\|_\infty \leqslant \|x\|_p \leqslant m^{1/p} \|x\|_\infty, \tag{13}$$

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty, \tag{14}$$

where $x \in \mathcal{R}^m$. Thus we obtain (see (12))

$$\begin{aligned} \|Z_p e_{j_1} - c_{j_1}\|_p &\geqslant \|Z_p e_{j_1} - c_{j_1}\|_\infty > \varepsilon + \min_{h \in \mathcal{R}^r} \|Ah + Z_p e_{j_1} - c_{j_1}\|_\infty \\ &\equiv \varepsilon + \|A\bar{h} + Z_p e_{j_1} - c_{j_1}\|_\infty \\ &\geqslant \varepsilon + m^{-1/p} \|A\bar{h} + Z_p e_{j_1} - c_{j_1}\|_p \geqslant \varepsilon + m^{-1/p} \|Z_p e_{j_1} - c_{j_1}\|_p \end{aligned}$$

because the j_1th column of X_p is the l_p-solution of (4). Hence we have (see (8) and (14))

$$\|Z_\infty e_j - c_j\|_\infty \geqslant \varepsilon + \|Z_\infty e_j - c_j\|_\infty,$$

which is a contradiction. Therefore, for $j = 1, ..., n$, the jth column of X_∞ is a Chebyshev solution of (9).

The same argument can be applied to prove that the columns of Y_∞^T are the Chebyshev solutions of (10), because the columns of Y_p^T are the l_p-solutions of (5). This completes the proof. ∎

We now prove that the matrices X_∞ and Y_∞ satisfy Conditions TS.

THEOREM 1. *Let X_∞ and Y_∞ satisfy* (8). *Then they satisfy Condition* TS.

Proof. We now prove that the jth column $x_j^{(\infty)}$ of X_∞ is a strict Chebyshev solution of (9) ($1 \leqslant j \leqslant n$). If the system (9) is consistent then $x_j^{(\infty)}$ is its solution and simultaneously it is a strict Chebyshev solution. We now assume that the system (9) is overdetermined.

Let

$$r_i(z) = \sum_{k=1}^{r} a_{ik} z_k + \sum_{l=1}^{s} y_{il}^{(\infty)} b_{lj} - c_{ij} \tag{15}$$

and let ρ_i, $\mathscr{R}_i$, $\mathscr{W}_i$ $(i = 1, ..., t)$ be the deviations, the characteristic sets, and the sets of the appropriate Chebyshev solutions of (9) defined in the construction of the strict Chebyshev solution of (9) (see [5], compare [7]). It is known that the strict Chebyshev solution of (9) is the unique solution of the consistent linear system

$$r_i(z) = \varepsilon(i)\, \rho_k, \qquad i \in \mathscr{R}_k \quad (k = 1, ..., t), \tag{16}$$

where $\varepsilon(i) = 1$ or -1 for all $z \in \mathscr{W}_k$. We now prove that $x_j^{(\infty)}$ satisfies the system (16). The vector $x_j^{(\infty)}$ is a Chebyshev solution of (9) (see Lemma 1), so we have

$$x_j^{(\infty)} \in \mathscr{W}_1 \subseteq \mathscr{F}_1, \tag{17}$$

where $\mathscr{F}_1$ is a linear manifold, $\mathscr{F}_1 = v^{(0)} + \ker(A^{(1)})$, $v^{(0)} \in \mathscr{W}_1$, and $A^{(1)}$ denotes the matrix of the system

$$r_i(z) = \varepsilon(i)\, \rho_1, \qquad i \in \mathscr{R}_1. \tag{18}$$

We recall that the linear manifold $\mathscr{F}_1$ is the set of solutions of (18), so $x_j^{(\infty)}$ satisfies (18). Therefore we have

$$r_{ij}^{(\infty)} = \varepsilon(i)\, \rho_1, \qquad i \in \mathscr{R}_1.$$

Suppose that $x_j^{(\infty)} \notin \mathscr{W}_2$. Thus we have (we denote $\mathscr{E} = \{1, ..., m\}$ and $\mathscr{G}_1 = \mathscr{E} \setminus \mathscr{R}_1$)

$$\rho_2 = \min_{z \in \mathscr{F}_1} \max_{i \in \mathscr{G}_1} |r_i(z)| < \max_{i \in \mathscr{G}_1} |r_{ij}^{(\infty)}|, \tag{19}$$

where $r_i(z)$ is determined by (15). Let $z \in \mathscr{F}_1$. Then $z - x_j^{(\infty)} \in \ker(A^{(1)})$ see (17)). Therefore we obtain

$$\rho_2 = \min_{h \in \ker(A^{(1)})} \max_{i \in \mathscr{G}_1} \left| \sum_{k=1}^{r} a_{ik} h_k + r_{ij}^{(\infty)} \right|, \tag{20}$$

where $h = (h_1, ..., h_r)^T \in \mathscr{R}^r$. We define

$$\psi_j(Z) = \max_{i \in \mathscr{G}_1} |z_{ij} - c_{ij}| - \min_{h \in \ker(A^{(1)})} \max_{i \in \mathscr{G}_1} \left| \sum_{k=1}^{r} a_{ik} h_k + z_{ij} - c_{ij} \right|,$$

where $Z=(z_{ij})\in \mathcal{M}_{mn}$. From (19) and (20) we have that there exists $\varepsilon>0$ such that

$$\psi_j(Z_\infty)>\varepsilon. \tag{21}$$

The function ψ_j is semi-lower continuous. Therefore there exists $\delta>0$ such that (see (11))

$$\psi_j(Z)>\varepsilon, \qquad \text{for} \quad Z\in \mathcal{M}_\delta. \tag{22}$$

Thus there exists p_0 such that (see (8)) $Z_p\in\mathcal{M}_\delta$ for $p>p_0$. Therefore we have (see (21))

$$\psi_j(Z_p)>\varepsilon, \qquad \text{for} \quad p>p_0.$$

Let

$$\max_{i\in\mathcal{S}_1} |f_i^{(p)}+r_{ij}^{(p)}| \equiv \min_{h\in\ker(A^{(1)})} \max_{i\in\mathcal{S}_1} \left|\sum_{k=1}^{r} a_{ik}h_k + r_{ij}^{(p)}\right|,$$

where

$$f_i^{(p)}=\sum_{k=1}^{r} a_{ik}h_k^{(p)}, \qquad i\in\mathcal{E}.$$

We have $f_i^{(p)}=0$ for $i\in\mathcal{R}_1$, because $h^{(p)}=(h_1^{(p)}, ..., h_r^{(p)})^T\in\ker(A^{(1)})$. Therefore we obtain

$$\sum_{i\in\mathcal{R}_1} |r_{ij}^{(p)}|^p = \sum_{i\in\mathcal{R}_1} |f_i^{(p)}+r_{ij}^{(p)}|^p \tag{23}$$

and consequently

$$\sum_{i\in\mathcal{S}_1} |r_{ij}^{(p)}|^p \leqslant \sum_{i\in\mathcal{S}_1} |f_i^{(p)}+r_{ij}^{(p)}|^p \tag{24}$$

since the jth column of X_p is the l_p-solution of system (4) and equality (23) holds. From (21)–(24) and (13) we obtain ($p>p_0$)

$$\begin{aligned}\left(\sum_{i\in\mathcal{S}_1} |r_{ij}^{(p)}|^p\right)^{1/p} &\geqslant \max_{i\in\mathcal{S}_1} |r_{ij}^{(p)}| > \varepsilon + \max_{i\in\mathcal{S}_1} |f_i^{(p)}+r_{ij}^{(p)}| \\ &\geqslant \varepsilon+(m-m_1)^{-1/p}\left(\sum_{i\in\mathcal{S}_1} |f_i^{(p)}+r_{ij}^{(p)}|^p\right)^{1/p} \\ &\geqslant \varepsilon+(m-m_1)^{-1/p}\left(\sum_{i\in\mathcal{S}_1} |r_{ij}^{(p)}|^p\right)^{1/p},\end{aligned} \tag{25}$$

where m_1 is the number of elements in the set $\mathscr{R}_1$. Let

$$\delta_j^* = \lim_{p \to \infty} \left(\sum_{i \in \mathscr{S}_1} |r_{ij}^{(p)}|^p \right)^{1/p}.$$

This limit exists because we have (8) and the relation (14) holds. Therefore we obtain (see (25)) $\delta_j^* \geqslant \varepsilon + \delta_j^*$, which is a contradiction. Hence the vector $x_j^{(\infty)}$ belongs to the set $\mathscr{W}_2$ and

$$r_{ij}^{(\infty)} = \varepsilon(i)\, \rho_2, \qquad \text{for} \quad i \in \mathscr{R}_2.$$

The same argument may be applied to prove that $x_j^{(\infty)} \in \mathscr{W}_k$ for $k = 3, 4, ..., t$. This implies that $x_j^{(\infty)}$ satisfies the system (16), so $x_j^{(\infty)}$ is a strict Chebyshev solution of (9).

Analogously we prove that for $i = 1, ..., m$, the ith column of Y_∞^T is a strict Chebyshev solution of (10), which completes the proof. ∎

In [13] we presented the following example.

EXAMPLE 1. Let $m = 6$, $n = 4$, $r = 2$, $s = 1$, and $B = [1, 1, 1, 1]$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1/8 \\ 1 & 1/4 \\ 1 & 1/2 \\ 1 & 3/4 \\ 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

The deviation ρ_1 for Eq. (1) is here equal to 0.98846154 and the characteristic set $\mathscr{R}_1$ has elements

$$\{(i, j)\colon |c_{ij}| = 1\} \cup \{(6, 2)\}.$$

It is easy to verify that the matrices $X = 0$ and $Y = 0$, $X \in \mathscr{M}_{rn}$, $Y \in \mathscr{M}_{ms}$, satisfy Condition TS. Unfortunately, they are not a Chebyshev solution of (1). This implies that Condition TS is not sufficient for X and Y to be a strict Chebyshev solution of (1) in the general case.

3. CONCLUSIONS

We now consider a particular case of the linear matrix equation (1). We assume

$$m = r + 1, \qquad n = s + 1, \qquad \text{rank}(A) = r, \qquad \text{rank}(B) = s. \qquad (26)$$

We recall that we have assumed that Condition (2) is not satisfied. We introduce auxiliary notation.

A_i—the matrix obtained from A by deleting the ith row;

B_j—the matrix obtained from B by deleting the jth column;

$$w_i = (-1)^i \det(A_i), \qquad u_j = (-1)^j \det(B_j);$$

$$\mathscr{S}_1 = \{i: w_i \neq 0\}, \qquad \mathscr{S}_2 = \{j: u_j \neq 0\};$$

$$w = (w_1, ..., w_{r+1})^T, \qquad u = (u_1, ..., u_{s+1})^T;$$

$$\gamma = -w^T Cu/\|wu^T\|_1, \qquad \text{where} \quad \|wu^T\|_1 = \|w\|_1 \, \|u\|_1 \quad \text{and} \quad \|w\|_1 = \sum_{i=1}^{r+1} |w_i|.$$

It is easy to verify the following corollary (compare Theorem 2.1 in [6]).

COROLLARY 1. *Let* $m = r + 1$ *and* $\operatorname{rank}(A) = r$. *Then the vector* $z \in \mathscr{R}^r$ *is a strict Chebyshev solution of the linear system* $Az = g$ *if and only if*

$$r_i(z) = \begin{cases} -\operatorname{sgn}(w_i)\, w^T g/\|w\|_1, & i \in \mathscr{S}_1, \\ 0, & i \notin \mathscr{S}_1, \end{cases}$$

where $r_i(z)$ *is the* i*th component of* $Az - g$.

We know that the set

$$\{(i, j): i \in \mathscr{S}_1, j \in \mathscr{S}_2\}$$

is the characteristic set of Eq. (1) (see [13]). Namely, the matrices X and Y are a Chebyshev solution of (1) under the assumptions (26) if and only if

$$r_{ij}(X; Y) = \operatorname{sgn}(w_i u_j)\, \gamma, \qquad \text{for} \quad i \in \mathscr{S}_1, \quad j \in \mathscr{S}_2 \tag{27}$$

and $|r_{ij}(X; Y)| \leqslant |\gamma|$ for other pairs of indices (i, j). We define the matrix $D = (d_{ij})$, $D \in \mathscr{M}_{r+1, s+1}$,

$$d_{ij} = \begin{cases} \operatorname{sgn}(w_i u_j)\, \gamma, & i \in \mathscr{S}_1, \quad j \in \mathscr{S}_2, \\ \alpha_{ij}, & i \notin \mathscr{S}_1 \quad \text{or} \quad j \notin \mathscr{S}_2, \end{cases}$$

where $|\alpha_{ij}| \leqslant |\gamma|$. In [12] we proved that the equation

$$AX + YB = C + D \tag{28}$$

is consistent and each of its solutions is a Chebyshev solution of (1) under the assumptions (26). If we take $\alpha_{ij} = 0$ for $i \notin \mathscr{S}_1$ or $j \notin \mathscr{S}_2$ then the solutions of (28) are the strict Chebyshev solutions of (1). Therefore we obtain the following corollary (compare Corollary 1).

COROLLARY 2. *The matrices X and Y are a strict Chebyshev solution of* (1) *under the assumptions* (26) *if and only if*

$$r_{ij}(X; Y) = \begin{cases} \operatorname{sgn}(w_i u_j)\,\gamma, & i \in \mathscr{S}_1, \quad j \in \mathscr{S}_2, \\ 0, & i \notin \mathscr{S}_1 \quad or \quad j \notin \mathscr{S}_2. \end{cases} \tag{29}$$

We now prove that X and Y are a strict Chebyshev solution of (1) under the assumptions (26) if and only if they satisfy Condition TS.

THEOREM 2. *Let the assumptions* (26) *be satisfied. Then the matrices X and Y are a strict Chebyshev solution of* (1) *if and only if they satisfy Condition* TS.

Proof. The first part of the theorem follows from Theorem 1. We now assume that X and Y satisfy Condition TS. Thus X and Y are a Chebyshev solution of (1) (see Theorem 3.1 in [13]). Since the jth column of X is a strict Chebyshev solution of (4) and the ith column of Y^T is a strict Chebyshev solution of (5), we have (see Corollary 1)

$$r_{ij}(X; Y) = 0, \qquad \text{for} \quad i \notin \mathscr{S}_1, \quad j = 1, ..., n,$$

$$r_{ij}(X; Y) = 0, \qquad \text{for} \quad j \notin \mathscr{S}_2, \quad i = 1, ..., m.$$

This implies

$$r_{ij}(X; Y) = 0, \qquad \text{for} \quad i \notin \mathscr{S}_1 \quad \text{or} \quad j \notin \mathscr{S}_2.$$

Moreover, the matrices X and Y satisfy the relations (27) because they are a Chebyshev solution of (1), so the relations (29) hold. This completes the proof. ∎

Now, let $r = s = 1$, $A \neq 0$, $B \neq 0$, and let m, n be arbitrary. If the matrices X and Y satisfy Condition TC then they are a Chebyshev solution of (1) (see [13]). After this, Condition TS is not suficient for X and Y to be a strict Chebyshev solution of (1). The following example shows it.

EXAMPLE 2. Let $m = n = 4$, $r = s = 1$, $A^T = B = [1, 1, 1, 1]$, and

$$C = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}.$$

The pairs $\hat{X} = \hat{Y}^T = [0, 0, 0, 0]$ and $\tilde{X} = [0, 0, 1, 1]$, $\tilde{Y} = [0, 0, -1, -1]^T$ are both Chebyshev solutions of (1) and they satisfy Condition TS. We have $R(\hat{X}; \hat{Y}) \neq R(\tilde{X}; \tilde{Y})$. The matrices $\hat{X}$ and $\hat{Y}$ are a strict Chebyshev solution of (1).

Methods for finding the strict Chebyshev solution of an overdetermined linear system are known (see [1, 2, 4, 6]). The direct application of these methods to the solution of (3) is not advisable because we cannot utilize its special form. Unfortunately, there does not exist an algorithm which utilizes the special form of (1).

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